Journal of Pure and Applied Mathematics: Advances and Applications
Volume 1, Number 1, 2009, Pages 59-82

UNKNOWN HETEROGENEITY IN NONCOOPERATIVE SOCIAL NETWORK FORMATION

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Abstract

A common assumption in the first literature of social network formation is homogeneity, in the sense that, on one hand, all decision makers conjecture that others receive information and establish links of equivalent value, on the other, links can fail independently of each other with the same probability. However, since empirical literature shows that ex-ante asymmetries across players arise quite naturally in reality, recent theoretical literature focuses on the role of heterogeneity.

In this work, a general kind of heterogeneity is considered in the framework of one sided two-way flow networks for situations in which agents do not have an a-priori opinion on the relative importance of benefits that each player conjectures to get from connections with the others. Two different models of network formation are here presented, corresponding to “relative" or “absolute" disutility of establishing direct connections (rd-networks and ad-networks), which are games with vector valued payoffs. It turns out that, for a certain class of parameters (low disutility), in the rd-networks the “two-way connectedness" and “no cycles”

2000 Mathematics Subject Classification: 90D30, 91A10, 91A43.

Keywords and phrases: network formation, heterogeneity, multicriteria games, equilibrium refinements.

Received December 5, 2008

properties characterize Pareto Nash equilibria while the center sponsored star is characterized by a refinement of Pareto Nash equilibrium called "ideal equilibrium". In the ad-networks, results are substantially different, in fact, on one hand Pareto Nash characterize only the "no cycles" property while simple examples show that a Pareto Nash can be disconnected. On the other hand, "two-way connectedness" is characterized by a generalization to multicriteria games of the "friendliness equilibrium" concept, meaning that altruistic motives increase the level of connectedness of the entire network.

1. Introduction

The processes of information diffusion within a group of individuals have attracted increasing interest in the economics literature and the analysis of these issues has developed in various directions. In the literature on social networks, individuals are a source of information for the others; moreover, they are identified with the vertices of a graph and create strategically relationships (links) within the others in such a way that the level of connectedness of the network determines the benefits of information they get. The basic assumption behind models of network formation is that establishing and maintaining connections with other individuals is costly. As a consequence, individuals limit the number or the intensity of their connections and then network structures develop from agents' comparison of disutility (costs) versus benefits of connection. In the seminal papers by Jackson and Wolinski [12] and by Dutta and Mutuswami [8] two-sided link formation has been investigated, in other words a situation in which two agents must agree on the decision to form a mutual link, but defection by one agent is sufficient to break the link; such notion of *pairwise stability* has been analyzed in a cooperative theoretic game framework. In Bala and Goyal [2], one-sided link formation has been studied, this is the case where an individual can form links with the others autonomously and incurring in the cost of connection. This situation allows a description in terms of non-cooperative theoretic game models and network stability has been studied in terms of Nash equilibrium concept and its refinements. In their paper, both the one-sided and two-sided flow of benefits (which correspond to directed and not directed graphs) are considered, and agents are supposed to be symmetric (homogeneous) and maximizing real valued payoff functions depending on two variables: the number of people (directly or indirectly) accessed and the number of links the agent forms himself. Moreover payoffs are assumed to be strictly increasing in the first variable and strictly decreasing in the second one. The first important result in Bala and Goyal [2] is that a *Nash network* is either empty or "minimally connected", i.e., there is a path between any couple of players and the delation of just one link renders the network disconnected. However, it turns out that there is a great number of Nash network so, as stated by Bala and Goyal, "*multiplicity of equilibria motivates an examination of a stronger equilibrium concept"*. They focus on the concept of *strict Nash equilibrium* which is characterized by the uniqueness of the best reply correspondences in equilibrium. They find out that, in the one-way flow model, the unique not empty (no links) strict Nash network is the *wheel*, that is, a connected network in which each player creates and receives one link. In the two-way flow model the unique not empty strict Nash network is the *center-sponsored star*, that is, a network in which one player, the center, forms links with all the other players. Star structures capture many real world situations in which there is a small average of shortest path length or in which "most" of the nodes are not neighbors of one another, but "most" nodes can be reached from every other by a "small" number of steps.

A common assumption in this first literature of network formation is homogeneity, in the sense that all decision makers conjecture that others provide and receive information of equivalent value or establish links with equivalent costs. Empirical literature shows that ex-ante asymmetries across players arise quite naturally in reality. For instance, agents might suppose the others are informed differently (information has different values) or differ in communication and social skills (forming links is cheaper for some individuals as compared to others). A first step in the investigation of the effects of agents heterogeneity in the network formation literature has been introduced by Johnson and Gilles [13] in which "spatial costs" are considered, representing geographical, social or individual differences and extending the two-sided network formation model. Other papers investigate the impact of ex-ante player heterogeneity on one-sided network formation, where heterogeneity is intended in terms of different values of information or different costs of connection while it is still implicitly assumed that players are able to compare *a*-priori the value the information coming from different opponents (see Galeotti et al. [10] for the two-way flow and Galeotti [9] for the one-way flow and references therein). Moreover in Haller and Sarangi [11] another point of view is taken into account: since links can fail independently of each other with a certain probability (probabilities of failure for all established links are identical in Bala and Goyal [3]), they introduce agent heterogeneity by allowing for the probability of link failure (or success) to differ across links.

In this paper we look at a different kind of heterogeneity in one-sided two way flow networks: agents are not able to compare *a*-priori the nature and the quality of information coming from the others. For instance, it may happen that players have information on different fundamentals or benefits of information are uncertain with unknown distributions which might be caused by the possibility of link failure with unknown distributions. This implies that there is no *a*-priori opinion on the relative importance of benefits that each player conjectures to get from connections with the others. In other words, it is studied the situation in which each player has an utility function associated to each other agent and which depends (increasingly) on the intensity of connection; moreover, these payoffs are not *a*-priori comparable. On the other hand, the cost of forming a link may differ qualitatively. In fact, on one hand, it can be relative to the benefits coming from the agent linked and we refer to these costs as the *rd-networks* (relative disutility networks). This is the case, for instance, of costs depending on the same uncertainty of the benefits, on the failure of the link (for instance phone calls) or, more generally, on the information coming from the agent linked. Consider, for example, medical test information: the outcome of a first test determines whether another test is needed or not and so on, therefore, the cost to have the information depend on the information itself. Another approach is to consider *ad-networks* (absolute disutility) in which costs are comparable each other but not comparable with benefits of connection (for instance, material costs whenever they do not depend on the information).

Hence, we consider agents endowed with vector valued payoff functions. In the *rd-networks*, payoffs have as many components as it is the number of each player's opponents. For every player i, the component of the payoff associated to player j is increasing in the level of connection between i and j and decreasing in the level of investment of player i in the direct link with player j. In the *ad-networks*, payoffs have as many components as it is the number of each player's opponents plus one. For every player i, the component of the payoff associated to player j is increasing in the level of connection between i and j, while the additional component represents costs and it is given by the (weighted) sum of levels of investment of player i in the direct links with the others. In this work, we consider players facing only binary choices: form links or not; i.e., the intensity of connection might be 0 or 1 and stability of network structures is analyzed in terms of Pareto Nash equilibria (also called Multicriteria Nash equilibria) and their refinements (see the seminal paper by Shapley [16], or also further results and references in Borm et al. [4]). It turns out that in the *rd-networks*, for a certain class of costs (low costs), the properties of "two-way connectedness" and "no cycles" (together called "minimally connectedness") characterize Pareto Nash equilibria (similarly to the classical two-way model in Bala and Goyal) and the center sponsored star is here completely characterized by a refinement of Pareto Nash equilibria called *ideal equilibrium* (see Voornerveld et al. [17]) in which the strategy of each player realizes the maximum of each component of his payoff function (also this result is in line with the corresponding result in Bala and Goyal [2]). While for high costs, it results that not empty networks are characterized by "no cycles" and the property that each link must provide also undirect connections. In the *ad-networks*, the results found are substantially different in fact, on one hand, Pareto Nash characterize only the "no cycles" property and simple examples show that a Pareto Nash can be disconnected. Moreover examples show that ideal equilibria and "*strong Nash"*-like refinements (Aumann [1]) are ineffective in *ad-networks*. However, it turns out that the "two-way connectedness" is characterized by a generalization to multicriteria games of the *friend-liness equilibrium* concept (studied in De Marco and Morgan [5, 6]) which is an extension of the so called *friendly behavior* property (defined and used by Rusinowska [15] for equilibrium selection in some 2-players bargaining models). Friendliness equilibria are based on a property of robustness of the equilibrium with respect to a particular class of deviations: a player is supposed to move away from the equilibrium even only to guarantee a better payoff to the others and feasible deviations are unilateral and only towards Nash equilibria. Finally, observe that friendliness equilibria have already been used for equilibrium selection in the classical homogeneous one sided two way flow networks in De Marco and Morgan [7], where, for instance, it has been proved that, whenever strict Nash equilibria do not exist then, the center sponsored star is an equilibrium in weakly dominated strategies but a friendliness equilibrium and the empty network is in weakly dominated strategies but not a friendliness equilibrium.

2. Multicriteria Games

Let $\Gamma = \{I; S_1, \ldots, S_n; J_1, \ldots, J_n\}$ be a multicriteria game, where $I = \{1, \ldots, n\}$ is set of players, S_i is the strategy set of each player i and $J_i : \prod_{j=1}^{n} S_j \to \mathbb{R}^{r(i)}$ is the vector-valued payoff function of player i, then:

Definition 2.1 (Shapley [16])**.** A strategy profile $s \in S$ of Γ is a *Pareto Nash Equilibrium* if, for each player i,

$$\nexists \hat{s}_i \in S_i \quad \text{s.t.} \quad J_i(\hat{s}_i, s_{-i}) - J_i(s_i, s_{-i}) \in \mathbb{R}_+^{r(i)} \setminus \{0\}.$$

This previous concept is one of the main generalizations of the Nash equilibrium concept (Nash [14]) to multicriteria games. We have also:

Definition 2.2. A strategy profile $s \in S$ of Γ is an *ideal equilibrium* if, for each player i:

$$J_i(s_i, s_{-i}) - J_i(\hat{s}_i, s_{-i}) \in \mathbb{R}_+^{r(i)} \quad \forall \hat{s}_i \in S_i.$$

Recall also that an equilibrium is said to be a strong Nash equilibrium (Aumann [1]) if no subset of players, taking the actions of the others fixed, can jointly deviate in a way that benefits positively all of them. Of course, also this concept might be extended to multicriteria games. To this purpose, if $s \in S$, denote $s_T = (s_j)_{j \in T}$ and $s_{-T} = (s_j)_{j \notin T}$. Then

Definition 2.3. A strategy profile s is a *strong Pareto Nash equilibrium* of Γ if for every subset of players $T \subseteq I$,

$$\nexists \hat{s}_T \in \prod_{j \in T} S_j \quad \text{s.t. } J_i(\hat{s}_T, s_{-T}) - J_i(s_T, s_{-T}) \in \mathbb{R}_+^{r(i)} \setminus \{0\} \quad \forall i \in T.$$

2.1. Friendliness property

Now a natural extension of the concept of *friendliness equilibrium* (De Marco and Morgan [5, 6]) is given for games with vector valued payoffs. Recall that a Nash equilibrium is said to be a friendliness equilibrium if no player has incentives to unilaterally deviate towards another Nash equilibrium when he maximizes his opponents' payoffs (*friendly behavior*). In other words, if an equilibrium is not a friendliness equilibrium, then there are incentives for unilateral deviations of the players caused by friendly behavior; moreover, even if every unilateral deviation is towards another element in the same component of Nash equilibria, simultaneous (but not coordinated) deviations of two or more players lead to a strategy profile which is not a Nash equilibrium.

Denote with $\mathcal{E}^P$ the set of Pareto Nash equilibria of the game. Let $K_i : S_{-i} \rightsquigarrow S_i$ be the set valued map defined by:

$$K_i(s_{-i}) = \{s_i \in S_i \mid (s_i, s_{-i}) \in \mathcal{E}^P\} \quad \text{for all } s_{-i} \in S_{-i}.$$

Then

Definition 2.4. A Pareto Nash equilibrium s^* is said to be a *friendliness Pareto Nash equilibrium ($\mathcal{FP}$ equilibrium)* of the game Γ if, for every player i, the following condition is satisfied:

$$\nexists \hat{s}_i \in K_i(s_{-i}) \quad \text{s.t.} \quad \begin{cases} J_h(\hat{s}_i, s_{-i}) - J_h(s_i, s_{-i}) \in \mathbb{R}_+^{r(h)}, \text{ for all } h \neq i \\ J_j(\hat{s}_i, s_{-i}) - J_j(s_i, s_{-i}) \in \mathbb{R}_+^{r(j)} \setminus \{0\}, \text{ for some } j \neq i. \end{cases}$$

3. The Network Formation Model

Following Bala and Goyal [2], we consider one-sided link formation networks. Let $I = \{1, \dots, n\}$, with $n \geq 3$, be the set of agents, where each agent is assumed to be a source of benefits for the others. Then each agent can improve his utility connecting with the others and incurring in some cost. A strategy for a player i is a $(n-1)$ dimensional vector

$$x_i = (x_{i,1}, \dots, x_{i,i-1}, x_{i,i+1}, \dots, x_{i,n})$$

with $x_{i,j} \in \{0, 1\}$, where $x_{i,j} = 1$ if i establishes a link with j and $x_{i,j} = 0$ otherwise; denote with X_i the strategy set of Player i and $X = X_1 \times \cdots \times X_n$.

A link between i and j can allow for either one-way or two-way flow of benefits. In the two-way flow of benefits $x_{i,j} = 1$ allows both i and j to access each other's benefit, while in the one-way flow $x_{i,j} = 1$ allows only Player i to access Player j's benefit. We consider only the two-way flow model; therefore a strategy profile x depicts one and only one undirected network. Denote $\mu(x_{i,j}) = \max\{x_{i,j}, x_{j,i}\} = \mu(x_{j,i})$, if $\mu(x_{i,j}) = 1$, then i and j are said to be *two-way directly connected* in the network x. Moreover, i and j are said to be *two-way connected* in the network x if there exists a *two-way path* between i and j in the network x, that is, a subset $P = \{j_1, \dots, j_m\} \subseteq I$ such that $i = j_1$, $j = j_m$ and $\mu(x_{j_h, j_{h+1}}) = 1$ for all $h = 1, \dots, m-1$ or equivalently:

$$\gamma_{i,j}^{P}(x) = \prod_{h=1}^{m-1} \mu(x_{j_h, j_{h+1}}) = 1.$$

Finally denote $\gamma_{i,j}(x) = \gamma_{j,i}(x) = 1$ if i and j are two-way connected and $\gamma_{i,j}(x) = \gamma_{j,i}(x) = 0$ otherwise.

It is possible to consider two different models: the relative disutility model and the absolute disutility model. In the first one, for each player i, we consider a $(n-1)$ dimensional vector payoff $\mathcal{R}_i : \prod_{j=1}^{n} X_j \to \mathbb{R}^{n-1}$, where, for every $j \in I \setminus \{i\}$, each component $\mathcal{R}_{i,j}(\cdot)$ of $\mathcal{R}_i(\cdot)$ is given by

$$\mathcal{R}_{i,j}(x) = \gamma_{i,j}(x) - c_{i,j}x_{i,j} \quad \text{for all } x \in X \tag{1}$$

and represents the utility for player i to be connected with j with intensity $\gamma_{i,j}(x)$ and effort $x_{i,j}$. The constant $c_{i,j}$ represents the relative disutility of player i to link j with respect to the utility for player i to be connected with player j. In this paper we consider only $c_{i,j} > 0$ for all i and j. We refer at the game

$$\Gamma_{rd} = \{I;\ X_1, \ldots, X_n;\ \mathcal{R}_1, \ldots, \mathcal{R}_n\}$$

as the *relative disutility* network game.

In the second model, for every player i we consider a vector valued function $\mathcal{A}_i : \prod_{j=1}^{n} X_j \to \mathbb{R}^n$, defined by

$$\mathcal{A}_i(x) = \left(\left(\gamma_{i,j}(x)\right)_{j \neq i},\ k_i(x)\right) \quad \text{for all } x \in X,$$

where each component $\gamma_{i,j}$ represents the utility to be connected with player j and the function $k_i : \prod_{j=1}^{n} X_j \to \mathbb{R}$ is given by

$$k_i(x) = -\sum_{j \neq i} c_{i,j} x_{i,j}, \tag{2}$$

where $-k_i$ represents the total absolute disutility of player i coming of establishing direct connections. So in this case the game

$$\Gamma_{ad} = \{I;\ X_1, \ldots, X_n;\ \mathcal{A}_1, \ldots, \mathcal{A}_n\}$$

is called *absolute disutility* network game.

Summarizing, it is assumed that every agent gives utility normalized to 1 to the others and these utilities are not comparable *a*-priori since they might be qualitatively different. Disutility linearly increase in the effort; in *rd-network* they are qualitatively comparable with the utilities but not comparable each other, in the *ad-network* they are comparable each other but not comparable with the utilities.

3.1. Homogeneous agents

In Bala and Goyal [2], for every player i, the payoff of each player is given by the function $\psi : \mathbb{N} \times \mathbb{N} \to \mathbb{R}$ which associates to (z_i, l_i) the term $\psi(z_i, l_i)$, where z_i is the number of players with whom Player i is (directly or indirectly) two-way connected (i included) and l_i is the number of players $j \neq i$ such that $x_{i,j} = 1$. It is assumed that ψ is strictly increasing in the first variable and strictly decreasing in the second one.

Obviously, z_i and l_i depend on the network formed and hence they are functions of the strategy profile, therefore setting $\psi(z_i(x_1, \ldots, x_n), l_i(x_1, \ldots, x_n)) = f_i(x_1, \ldots, x_n)$, it is possible to consider the following game of network formation:

$$\Gamma^T = \{I; \{X_i\}_i; \{f_i\}_i\}.$$

Recall that

Definition 3.1. Given a network x, then a *cycle* in x is a subset of players $\{j_1, \ldots, j_q\} \subseteq I$ such that

$$\mu(x_{j_1, j_2}) = \cdots = \mu(x_{j_{q-1}, j_q}) = \mu(x_{j_q, j_1}) = 1.$$

Moreover, a network x is said to be the empty network if $x_{i,j} = 0$ for all $i, j \in I$ with $i \neq j$.

Then

Proposition 3.2 [Bala and Goyal [2]]**.** *If x is a Nash equilibrium of Γ^T, then the corresponding network is either empty or satisfies the following:*

i) *Every couple of players (i, j) is two-way connected in x.*

ii) *There does not exist a cycle in x.*

The previous result shows that a great variety of networks can be implemented by Nash equilibria of the corresponding game, however some network structures play a predominant role:

Definition 3.3. A network x is said to be a *center-sponsored star* if there exists $i \in I$ such that $x_{i,j} = 1$ for all j and $x_{j,h} = 0$ for all $j \neq i$ and for all h.

Proposition 3.4 [Bala and Goyal [2]]**.** *Let x be a strict Nash equilibrium x of Γ^T, then x is either a center-sponsored star or the empty network. If x is a center-sponsored star, then it is a strict Nash equilibrium if and only if $\psi(n, n-1) > \psi(w+1, w)$ for all $w \in \{0, \dots, n-2\}$. If x is the empty network, then it is a strict Nash equilibrium if and only if $\psi(1, 0) > \psi(w+1, w)$ for all $w \in \{1, \dots, n-1\}$.*

4. Heterogeneity: rd-networks

Proposition 4.1. *Assume that $0 < c_{i,j}$ for all i and for all $j \neq i$. If x a Pareto Nash equilibrium of Γ_{rd}, then there does not exist a cycle in x.*

Proof. Suppose there exist a cycle, i.e., a subset of players $\{j_1, \dots, j_q\} \subseteq I$ such that

$$\mu(x_{j_1,j_2}) = \cdots = \mu(x_{j_{q-1},j_q}) = \mu(x_{j_q,j_1}) = 1.$$

Without loss of generality, there exist $j_\alpha, j_\beta \in \{j_1, \dots, j_q\}$ such that $x_{j_\alpha,j_\beta} = 1$, then player j_α may deviate, choosing the strategy $\overline{x}_{j_\alpha}$, where $\overline{x}_{j_\alpha,k} = x_{j_\alpha,k}$ for all $k \in I \setminus \{j_\alpha, j_\beta\}$ and $\overline{x}_{j_\alpha,j_\beta} = 0$. Since $\{j_1, \dots, j_q\}$ is a cycle in x, then $\gamma_{j_\alpha,j_\beta}(\overline{x}_{j_\alpha}, x_{-j_\alpha}) = 1$ and

$$\mathcal{R}_{j_\alpha,j_\beta}(\overline{x}_{j_\alpha}, x_{-j_\alpha}) = 1 > 1 - c_{j_\alpha,j_\beta} x_{j_\alpha,j_\beta} = \mathcal{R}_{j_\alpha,j_\beta}(x_i, x_{-i}).$$

Moreover $\mathcal{R}_{j_\alpha,h}(\overline{x}_{j_\alpha}, x_{-j_\alpha}) = \mathcal{R}_{j_\alpha,h}(x_{j_\alpha}, x_{-j_\alpha})$ for every $h \in I \setminus \{j_\alpha, j_\beta\}$, hence x is not a Pareto Nash and we get a contradiction. So there are no cycles in x.

Note that in this previous proposition there are no restrictions on the parameters $c_{i,j}$ (just assumed to be positive since they represent disutilities). Below we provide results depending on the parameters $c_{i,j}$. Firstly low costs are considered; it turns out that in this case, the characterization of Pareto Nash equilibria is in line with the characterizations of Nash equilibria in the classical models of network formation.

Proposition 4.2. *Assume that* $0 < c_{i,j} < 1$ *for all i and for all* $j \neq i$, *then a Pareto Nash equilibrium x of* Γ_{rd} *is a two-way connected network.*

Proof. Let x be a Pareto Nash equilibrium of Γ_{rd}. Suppose there exist player i and player j such that $\gamma_{i,j}(x) = 0$, then $\mathcal{R}_{i,j}(x) = 0$. Consider another strategy $\overline{x}_i$ of player i in which $\overline{x}_{i,j} = 1$ while $\overline{x}_{i,h} = x_{i,h}$ for all $h \in I \setminus \{i, j\}$. It follows that

$$\mathcal{R}_{i,j}(\overline{x}_i, x_{-i}) = 1 - c_{i,j}\overline{x}_{i,j} = 1 - c_{i,j} > 0 = \mathcal{R}_{i,j}(x_i, x_{-i}).$$

For every other h with $h \neq j$, $h \neq i$ it results that

$$\begin{cases} \gamma_{i,h}(x_i, x_{-i}) = 0 \\ \gamma_{j,h}(x_i, x_{-i}) = 1 \end{cases} \Rightarrow \mathcal{R}_{i,h}(\overline{x}_i, x_{-i}) > \mathcal{R}_{i,h}(x_i, x_{-i})$$

while $\mathcal{R}_{i,h}(\overline{x}_i, x_{-i}) = \mathcal{R}_{i,h}(x_i, x_{-i})$ if $\gamma_{i,h}(x_i, x_{-i}) = \gamma_{j,h}(x_i, x_{-i}) = 0$ or if $\gamma_{i,h}(x_i, x_{-i}) = \gamma_{j,h}(x_i, x_{-i}) = 1$. Therefore $\overline{x}_i$ Pareto dominates x_i and we get a contradiction. Hence x is two-way connected.

Proposition 4.3. *Assume that* $0 < c_{i,j} < 1$ *for all i and for all* $j \neq i$. *Let x be a network satisfying*

(i) *Every couple of players* (i, j) *is two-way connected.*

(ii) *There does not exist a cycle in x.*

Then, x is a Pareto Nash equilibrium of Γ_{rd}.

Proof. Assume x is a network satisfying (i), (ii) and suppose it is not a Pareto Nash equilibrium. Therefore there exists a player i and a strategy $\bar{x}_i$ such that

$$\mathcal{R}_i(\bar{x}_i, x_{-i}) - \mathcal{R}_i(x_i, x_{-i}) \in \mathbb{R}_+^{n-1} \setminus \{0\}. \tag{3}$$

Hence there exists $j \in I \setminus \{i\}$ such that $\mathcal{R}_{i,j}(\bar{x}_i, x_{-i}) > \mathcal{R}_{i,j}(x_i, x_{-i})$. In light of the assumption (i) in (3), it follows that $\gamma_{i,j}(x_i, x_{-i}) = 1$ and, then $\mathcal{R}_{i,j}(x_i, x_{-i}) = 1$ or $\mathcal{R}_{i,j}(x_i, x_{-i}) = 1 - c_{i,j}$. If $\mathcal{R}_{i,j}(x_i, x_{-i}) = 1$ we clearly get a contradiction. If $\mathcal{R}_{i,j}(x_i, x_{-i}) = 1 - c_{i,j}x_{i,j}$, with $x_{i,j} = 1$, then we get

$$\mathcal{R}_{i,j}(\bar{x}_i, x_{-i}) = \gamma_{i,j}(\bar{x}_i, x_{-i}) - c_{i,j}\bar{x}_{i,j} > 1 - c_{i,j}x_{i,j} = \mathcal{R}_{i,j}(x_i, x_{-i}). \tag{4}$$

If $\gamma_{i,j}(\bar{x}_i, x_{-i}) = 0$, then $\bar{x}_{i,j} = 0$ and $\gamma_{i,j}(\bar{x}_i, x_{-i}) - c_{i,j}\bar{x}_{i,j} < 1 - c_{i,j}$, which contradicts (4). Hence $\gamma_{i,j}(\bar{x}_i, x_{-i}) = 1$, therefore,

$$\mathcal{R}_{i,j}(\bar{x}_i, x_{-i}) > \mathcal{R}_{i,j}(x_i, x_{-i}) \Leftrightarrow 1 - c_{i,j}\bar{x}_{i,j} > 1 - c_{i,j}x_{i,j} \Leftrightarrow \bar{x}_{i,j} < x_{i,j} \Leftrightarrow \bar{x}_{i,j} = 0.$$

Moreover, since $\gamma_{i,j}(\bar{x}_i, x_{-i}) = 1$, let $P = \{i = j_0, j_1, \dots, j_m, j_{m+1} = j\}$ be a path between player i and player j in the network $(\bar{x}_i, x_{-i})$, that is, $\gamma^P_{i,j}(\bar{x}_i, x_{-i}) = \gamma_{i,j}(\bar{x}_i, x_{-i})$. It has to be that $\gamma^P_{i,j}(x_i, x_{-i}) = 0$ otherwise there exists a cycle in the network x, in fact it would result that $\mu(x_{i,j_1}) = \mu(x_{j_1,j_2}) = \cdots = \mu(x_{j_m,j}) = \mu(x_{j,i}) = 1$. Denote $\bar{x} = (\bar{x}_i, x_{-i})$, then obviously it follows that $\mu(x_{j_1,j_2}) = \mu(\bar{x}_{j_1,j_2}) = \cdots = \mu(x_{j_m,j}) = \mu(\bar{x}_{j_m,j}) = 1$. Therefore $\gamma^P_{i,j}(x_i, x_{-i}) = 0$ and, then it follows that $\mu(\bar{x}_{i,j_1}) = 1 > \mu(x_{i,j_1}) = 0$. Then it has to be that $x_{i,j_1} = 0$ and $\bar{x}_{i,j_1} = 1$, but in light of the assumption (i) it results that $\gamma_{i,j_1}(x) = 1$ so $\mathcal{R}_{i,j_1}(x_i, x_{-i}) = 1$, and

$$\mathcal{R}_{i,j_1}(\bar{x}_i, x_{-i}) = 1 - c_{i,j_1}\bar{x}_{i,j_1} < 1 = \mathcal{R}_{i,j_1}(x_i, x_{-i}).$$

In light of this inequality, (3) does not hold, hence x is a Pareto Nash equilibrium.

Remark 4.4. Propositions (4.1), (4.2) and (4.3) guarantee that whenever $0 < c_{i,j} < 1$ for all i and $j \neq i$, the "no cycles" and the two-way connectedness properties are necessary and sufficient conditions for Pareto Nash equilibria.

The next two results show the relation between Pareto Nash equilibria and the property that each direct link in a network provides also undirect connections in the case of high costs.

Proposition 4.5. *Assume that* $c_{i,j} > 1$ *for all i and for all* $j \neq i$. *If a network x is a Pareto Nash equilibrium, then x satisfies the following property*

$$x_{i,j} = 1 \Rightarrow \exists k \in I \setminus \{i, j\} \text{ such that } \mu(x_{j,k}) = 1. \tag{5}$$

Proof. Let x be a Pareto Nash equilibrium. Suppose that $x_{i,j} = 1$ and $\mu(x_{j,k}) = 0$ for all $k \in I \setminus \{i, j\}$. Consider another strategy for player i, $\bar{x}_i$ such that $\bar{x}_{i,j} = 0$ and $\bar{x}_{i,k} = x_{i,k}$ for all $k \in I \setminus \{i, j\}$. It results that

$\mathcal{R}_{i,j}(\bar{x}_i, x_{-i}) = 0 > 1 - c_{i,j} = \mathcal{R}_{i,j}(x_i, x_{-i})$ and $\mathcal{R}_{i,k}(\bar{x}_i, x_{-i}) = \mathcal{R}_{i,k}(x_i, x_{-i})\ \forall k \in I \setminus \{i, j\}$.

Therefore

$$\mathcal{R}_i(\bar{x}_i, x_{-i}) - \mathcal{R}_i(x_i, x_{-i}) \in \mathbb{R}_+^{n-1} \setminus \{0\}$$

and x is not a Pareto Nash equilibrium. Hence the contradiction which implies that

$x_{i,j} = 1 \Rightarrow \exists k \in I \setminus \{i, j\}$ such that $\mu(x_{j,k}) = 1$.

Proposition 4.6. *Assume that* $c_{i,j} > 1$ *for all i and for all* $j \neq i$. *Let x be a network satisfying*

(i) $x_{i,j} = 1 \Rightarrow \exists k \in I \setminus \{i, j\}$ *such that* $\mu(x_{j,k}) = 1$.

(ii) *There does not exist a cycle in x.*

Then, x is a Pareto Nash equilibrium of Γ_{rd}.

Proof. Let x be a network satisfying property (i), (ii). Suppose x is not a Pareto Nash equilibrium, then there exists a player i and a strategy $\bar{x}_i$ such that

$$\mathcal{R}_i(\bar{x}_i, x_{-i}) - \mathcal{R}_i(x_i, x_{-i}) \in \mathbb{R}_+^{n-1} \setminus \{0\}.$$

Note firstly that $\bar{x}_i$ cannot be obtained from x_i by forming new links, in fact, if $\bar{x}_{i,j} = 1 > 0 = x_{i,j}$, then $\mathcal{R}_{i,j}(\bar{x}_i, x_{-i}) = 1 - c_{i,j} < 0 = \mathcal{R}_{i,j}(x_i, x_{-i})$. Therefore $\bar{x}_i$ is obtained from x_i only by delating links; let j be such that $x_{i,j} = 1 > 0 = \bar{x}_{i,j}$. In light of the assumptions, there exists k such that $\mu(x_{j,k}) = 1$, then $\mathcal{R}_{i,k}(x_i, x_{-i}) = 1$; in fact if it was $\mathcal{R}_{i,k}(x_i, x_{-i}) = 1 - c_{i,k}$, then $\{i, j, k\}$ is a cycle in x. Moreover $\gamma_{i,k}(\bar{x}_i, x_{-i}) = 0$, in fact $\gamma_{i,k}(\bar{x}_i, x_{-i}) = 1$ implies there exists a path $P \subset I$ connecting i and k in $(\bar{x}_i, x_{-i})$ but since there are no new links in $\bar{x}_i$, then $P \cup \{j\}$ is a cycle in x which contradicts the assumptions. Therefore

$$\mathcal{R}_{i,k}(\bar{x}_i, x_{-i}) = 0 < 1 = \mathcal{R}_{i,k}(x_i, x_{-i})$$

and therefore

$$\mathcal{R}_i(\bar{x}_i, x_{-i}) - \mathcal{R}_i(x_i, x_{-i}) \notin \mathbb{R}_+^{n-1} \setminus \{0\}.$$

Hence the contradiction and x is a Pareto Nash equilibrium of Γ_{rd}.

Remark 4.7. Propositions (4.1), (4.5) and (4.6) guarantee that whenever $c_{i,j} > 1$ for all i and $j \neq i$, "no cycles" and the property defined in equation (5) are necessary and sufficient conditions for Pareto Nash equilibria. Moreover, note that "no cycles" guarantees that if in a network x, in which $x_{i,j} = 1$ and $\mu(x_{j,k}) = 1$, player i delate the link with j, then i and k will be no longer two-way connected; in other words if $\bar{x}$ is another network which differs from x only in the component $\bar{x}_{i,j} = 0 \neq x_{i,j}$, then $\gamma_{i,k}(\bar{x}) = 0$. Therefore "no cycles" and the property defined in equation (5) guarantee that each direct link is necessary to obtain further undirect connections.

The properties of the empty network are summarized in the following proposition.

Proposition 4.8. *The empty network is an ideal equilibrium of* Γ_{rd} *if and only if* $c_{i,j} \geq 1$ *for all* i *and for all* $j \neq i$. *If the inequality are all strict the empty network is the unique ideal equilibrium of* Γ_{rd}. *If* $\exists i, j \in I$, *with* $i \neq j$ *and* $c_{i,j} < 1$, *then the empty network is not a Pareto Nash equilibrium of* Γ_{rd}.

Proof. Trivially $x_{i,j} = 0$ for all $j \neq i$ maximizes $\mathcal{R}_{i,j}(\cdot, x_{-i})$ for all $j \neq i$ if and only if $c_{i,j} \geq 1$. Moreover if $c_{i,j} > 1$, then $x_{i,j} = 0$ is the unique maximum point for $\mathcal{R}_{i,j}(\cdot, x_{-i})$. Finally, if $\exists i, j \in I$, with $i \neq j$ and $c_{i,j} < 1$, then player i has incentives to deviate from the empty network, for example by choosing a strategy $\overline{x}_i$ defined by $\overline{x}_{i,j} = 1$ and $\overline{x}_{i,t} = x_{i,t}$ for all $t \in I \setminus \{i, j\}$, in this case the payoff of player i would be $\mathcal{R}_{i,j}(\overline{x}_i, x_{-i}) = 1 - c_{i,j} > 0 = \mathcal{R}_{i,j}(x_i, x_{-i})$ and $\mathcal{R}_{i,t}(\overline{x}_i, x_{-i}) = 0 = \mathcal{R}_{i,t}(x_i, x_{-i})$ for all $t \neq i, j$ which means that $\mathcal{R}_i(\overline{x}_i, x_{-i}) - \mathcal{R}_i(x_i, x_{-i}) \in \mathbb{R}_+^{n-1} \setminus \{0\}$.

A characterization of ideal equilibria in terms of star structures is given below. This result is in line with the characterization of strict Nash equilibria in the classical models of network formation.

Proposition 4.9. *Assume that* $0 < c_{i,j} < 1$ *for all i and for all* $j \neq i$, *then a network x is an ideal equilibrium of* Γ_{rd} *if and only if x is a center sponsored star.*

Proof. Suppose x is a center sponsored star and i the center of the star, then

$$\mathcal{R}_{i,j}(x_i, x_{-i}) = 1 - c_{i,j} \geq \overline{x}_{i,j} - c_{i,j}\overline{x}_{i,j} = \mathcal{R}_{i,j}(\overline{x}_i, x_{-i}) \quad \forall \overline{x}_i \in X_i, \forall j \in I \setminus \{i\}.$$

For any other player $j \in I \setminus \{i\}$,

$$\mathcal{R}_{j,k}(x_j, x_{-j}) = 1 \geq 1 - c_{j,k}\overline{x}_{j,k} = \mathcal{R}_{j,k}(\overline{x}_j, x_{-j}) \quad \forall \overline{x}_j \in X_j, \text{ and } \forall k \in I \setminus \{j\}.$$

Hence the implication immediately follows. Conversely, assume that x is an ideal equilibrium of Γ_{rd}, then it is also a Pareto Nash equilibrium and satisfy the previous Propositions (4.1) and (4.2). In particular x it is not empty and then consider $i, j \in I$ such that $x_{i,j} = 1$. Suppose there exists $h \in I \setminus \{i, j\}$ such that $x_{i,h} = 0$. From the assumptions players j and h are two-way connected in the network x. Consider another strategy $\bar{x}_i$ for player i, defined by $\bar{x}_{i,j} = 0, \bar{x}_{i,h} = 1$ and $\bar{x}_{i,t} = x_{i,t}$ for all $t \in I \setminus \{i, j, h\}$, in this case the payoff of player i would be

$$\mathcal{R}_{i,j}(\bar{x}_i, x_{-i}) = 1 > 1 - c_{i,j} = \mathcal{R}_{i,j}(x_i, x_{-i})$$

which means that

$$\mathcal{R}_i(x_i, x_{-i}) - \mathcal{R}_i(\bar{x}_i, x_{-i}) \notin \mathbb{R}_+^{n-1}$$

and x is not an ideal equilibrium. From the contradiction it follows that

$$x_{i,j} = 1 \Rightarrow x_{i,h} = 1 \text{ for all } h \in I \setminus \{i\}. \tag{6}$$

Moreover, not emptiness of x implies that there exists a player i satisfying (6). Finally, since cycles do not exist, $x_{i,j} = 1$ implies $x_{j,i} = 0$. While, if $x_{h,k} = 1$ for some $h, k \neq i$, then there exists a cycle in $x : \mu(x_{i,h}) = \mu(x_{h,k}) = \mu(x_{k,i}) = 1$, so $x_{h,k} = 0$ for all $h, k \neq i$. Therefore x is a center sponsored star.

Remark 4.10. The "if" part of the previous proposition (center sponsored star are ideal equilibria) holds true even under the weaker assumptions: $0 < c_{i,j} \leq 1$ for all i and j. However it is easy to check that whenever for the center of the star, say player i, $c_{i,j} > 1$ for some j, then the center sponsored star is not even a Pareto Nash equilibrium. We show below that other star network structures might be Pareto Nash equilibria.

Recall that:

Definition 4.11. A network x is said to be a *periphery-sponsored star* if there exists $i \in I$ such that, for all $j \neq i$, $x_{j,i} = 1$ and $x_{j,h} = 0$ for all $h \neq i$, while $x_{i,k} = 0$ for all $k \neq i$.

It easily follows from Proposition (4.6) that

Corollary 4.12. *Assume that* $c_{i,j} > 1$ *for all* i *and for all* $j \neq i$. *Then, a periphery sponsored star is a Pareto Nash equilibrium of* Γ_{rd}.

Note also that "mixed-sponsored" star networks might be equilibria for suitable constants $c_{i,j}$.

5. Heterogeneity: ad-networks

Differently from the *rd-networks*, in a *ad-network* a Pareto Nash equilibrium is not necessarily two-way connected even when $c_{i,j} < 1$ for all i and j as shown in the following example:

Example 5.1. Let $I = \{1, 2, 3\}$, $c_{i,j} = 0.1$ for all i and j and x be such that $x_{1,2} = 1$ and all the other $x_{i,j} = 0$ for $i \neq 1$ and $j \neq 2$. It is easy to check that it is a Pareto Nash equilibrium of Γ_{ad} even if it is not two-way connected.

It is easy to check that:

Proposition 5.2. *The empty network is a Pareto Nash equilibrium of* Γ_{ad}.

Proof. It obviously follows from the fact that the no link strategy maximizes each function $k_i(\cdot, x_{-i})$ for every strategy profile x_{-i}.

A characterization of not empty Pareto Nash networks is given below:

Proposition 5.3. *If a not empty network* x *is a Pareto Nash equilibrium of* Γ_{ad}, *then* x *does not have cycles. Conversely, if* $c_{i,j} = c_i$ *for all* i *and for all* $j \neq i$, *and if the not empty network* x *does not have cycles, then* x *is a Pareto Nash equilibrium of* Γ_{ad}.

Proof. Let x be a not empty Pareto Nash equilibrium of Γ_{ad}. Suppose there exists a cycle, i.e., a subset of players $\{j_1, \ldots, j_q\} \subseteq I$ such that

$$\mu(x_{j_1,j_2}) = \cdots = \mu(x_{j_{q-1},j_q}) = \mu(x_{j_q,j_1}) = 1.$$

Without loss of generality, there exist $j_\alpha, j_\beta \in \{j_1, \dots, j_q\}$ such that $x_{j_\alpha, j_\beta} = 1$. Assume player j_α deviate by delating the link with j_β, i.e., by choosing the strategy $\bar{x}_{j_\alpha}$, where $\bar{x}_{j_\alpha, k} = x_{j_\alpha, k}$ for all $k \in I \setminus \{j_\alpha, j_\beta\}$ and $\bar{x}_{j_\alpha, j_\beta} = 0$. Since $\{j_1, \dots, j_q\}$ is a cycle in x, then $\gamma_{j_\alpha, j_\beta}(\bar{x}_{j_\alpha}, x_{-j_\alpha}) = 1 = \gamma_{j_\alpha, j_\beta}(x_{j_\alpha}, x_{-j_\alpha})$. Moreover $\gamma_{j_\alpha, h}(\bar{x}_{j_\alpha}, x_{-j_\alpha}) = \gamma_{j_\alpha, h}(x_{j_\alpha}, x_{-j_\alpha})$ for every $h \in I \setminus \{j_\alpha, j_\beta\}$. Finally, $k_{j_\alpha}(\bar{x}_{j_\alpha}, x_{-j_\alpha}) = k_{j_\alpha}(x_{j_\alpha}, x_{-j_\alpha}) + c_{j_\alpha, j_\beta} > k_{j_\alpha}(x_{j_\alpha}, x_{-j_\alpha})$ therefore

$$\mathcal{A}_{j_\alpha}(\bar{x}_{j_\alpha}, x_{-j_\alpha}) - \mathcal{A}_{j_\alpha}(x_{j_\alpha}, x_{-j_\alpha}) \in \mathbb{R}^n_+ \setminus \{0\}$$

hence x is not a Pareto Nash and we get a contradiction. Therefore x does not have cycles.

Conversely, assume that $c_{i,j} = c_i$ for all i and for all $j \neq i$ and let x be a not empty network without cycles. Suppose x is not a Pareto Nash equilibrium, then there exists a player i and a strategy $\bar{x}_i$ which improves x_i (with respect to Pareto dominance for the function $\mathcal{A}_i$). Now we show that in the strategy $\bar{x}_i$ there do not exist new links:

$$\nexists k \in I \setminus \{i\} \quad \text{such that } \bar{x}_{i,k} = 1 \text{ and } x_{i,k}(x_i, x_{-i}) = 0.$$

In fact, let $k \in I \setminus \{i\}$ be such that $\bar{x}_{i,k} = 1$ and $x_{i,k}(x_i, x_{-i}) = 0$. If in $\bar{x}_i$ no links are delated, that is, $\nexists h \in I \setminus \{i, k\}$ such that $\bar{x}_{i,h} = 0$ and $x_{i,h} = 1$, then $k_i(\bar{x}_i, x_{-i}) < k_i(x_i, x_{-i})$ and $\bar{x}_i$ does not improve x_i. Suppose there exists $j \in I \setminus \{i, k\}$ such that $\bar{x}_{i,j} = 0$ and $x_{i,j} = 1$. If $\gamma_{i,j}(\bar{x}_i, x_{-i}) = 0 < 1 = \gamma_{i,j}(x_i, x_{-i})$, then $\bar{x}_i$ does not improve x_i. If $\gamma_{i,j}(\bar{x}_i, x_{-i}) = 1 = \gamma_{i,j}(x_i, x_{-i})$, then there has to be a two-way path $P \subset I$ between i and j in the network $(\bar{x}_i, x_{-i})$; since P is not a cycle in (x_i, x_{-i}), then the delation of the link $x_{i,j}$ implies that player i has formed at least another link in $\bar{x}$, that is, there exists $l \neq k$ such that $\bar{x}_{i,l} = 1$ and $x_{i,l} = 0$ and hence $k_i(\bar{x}_i, x_{-i}) < k_i(x_i, x_{-i})$ which implies again that $\bar{x}_i$ does not improve x_i. So, if $\bar{x}_i$ improves x_i, $\nexists k \in I \setminus \{i\}$ such that $\bar{x}_{i,k} = 1$ and $x_{i,k}(x_i, x_{-i}) = 0$. Hence $\bar{x}_i$ is obtained from x_i only by delating links. Let $h \neq i$ be such that $\bar{x}_{i,h} = 0$ and $x_{i,h} = 1$; since there are no cycles in x it has to be that $\gamma_{i,j}(\bar{x}_i, x_{-i}) = 0 < 1 = \gamma_{i,j}(x_i, x_{-i})$, then $\bar{x}_i$ does not improve x_i and hence we get a contradiction. Hence x is a Pareto Nash equilibrium of Γ_{ad}.

As we have seen, the concept of Pareto Nash equilibrium is not able to guarantee connectedness of the corresponding network. However, we show below that $\mathcal{FP}$ equilibria achieve this task.

Proposition 5.4. *If a network x is a friendliness Pareto Nash equilibrium of Γ_{ad}, then it is two-way connected and does not have cycles. Conversely, if $c_{i,j} = c_i$ for all i and for all $j \neq i$, and if the not empty network x is two-way connected and does not have cycles, then x is a friendliness Pareto Nash equilibrium of Γ_{ad}.*

Proof. Let x be a $\mathcal{FP}$ equilibrium, then, in light of the previous proposition, it does not have cycles. Suppose x is not two-way connected, that is, there exist two players, say i and j which are not two-way connected, i.e., $\gamma_{i,j}(x) = 0$. Let x_i be the equilibrium strategy of player i, then it obviously follows that $x_{i,j} = 0$. Consider a new strategy of player i, $\bar{x}_i$ given by $\bar{x}_{i,h} = x_{i,h} \forall h \in I \setminus \{i, j\}$, $\bar{x}_{i,j} = 1$. For player i, it follows

$$\gamma_{i,j}(\bar{x}_i, x_{-i}) = 1 > 0 = \gamma_{i,j}(x_i, x_{-i})$$

so $\bar{x}_i$ is a best reply (with respect to Pareto dominance for the function $\mathcal{A}_i$) to x_{-i}. Moreover, $\bar{x}_i \in K_i(x_{-i})$ since, for every player $l \neq i$, the strategy x_l is a best reply to the strategy profile $((x_h)_{h \neq i, l}, \bar{x}_i)$ of players in $I \setminus \{l\}$. In fact, on one hand, x_l cannot be improved (with respect to Pareto dominance for the function $\mathcal{A}_l$) only by the formation of new links, say with player k, as in this case the new strategy $\tilde{x}_l$ gives higher costs surely, i.e., we have $k_l(\tilde{x}_l, (x_h)_{h \neq i, l}, \bar{x}_i) \leq k_l(x_l, (x_h)_{h \neq i, l}, \bar{x}_i) - c_{l,k} <$

$k_l(x_l, (x_h)_{h \neq i, l}, \bar{x}_i)$. On the other hand, since x does not have cycles and $\gamma_{i,j}(x) = 0$ then $((x_h)_{h \neq i}, \bar{x}_i)$ does not have cycles, otherwise it would be $\gamma_{i,j}(x) = 1$. Hence x_l cannot be improved by the delation of connections, say with player k, as in this case the new strategy $\bar{x}_l$ gives to player l a lower payoff (equal to 0) corresponding to player k, i.e., $\gamma_{l,k}(\bar{x}_l, (x_h)_{h \neq i, l} \bar{x}_i) = 0 < 1 = \gamma_{l,k}(x_l, (x_h)_{h \neq i, l}, \bar{x}_i)$. Therefore $\bar{x}_i \in K_i(x_{-i})$.

It follows that, for player j:

$$\gamma_{j,i}(\bar{x}_i, x_{-i}) = 1 > 0 = \gamma_{j,i}(x_i, x_{-i}).$$

Moreover, for $h \in I \setminus \{i, j\}$

$$\begin{cases} \gamma_{j,h}(x_i, x_{-i}) = 0 \\ \gamma_{i,h}(x_i, x_{-i}) = 1 \end{cases} \Rightarrow \gamma_{j,h}(\bar{x}_i, x_{-i}) = 1 > \gamma_{j,h}(x_i, x_{-i})$$

while $\gamma_{j,h}(\bar{x}_i, x_{-i}) = \gamma_{j,h}(x_i, x_{-i}) = 0$ or $\gamma_{j,h}(\bar{x}_i, x_{-i}) = \gamma_{j,h}(x_i, x_{-i}) = 1$ otherwise.

For every player $k \in I \setminus \{i, j\}$ and for $h \in I \setminus \{k\}$

$$\begin{cases} \gamma_{k,h}(x_i, x_{-i}) = 0 \\ \gamma_{k,j}(x_i, x_{-i}) = 1 \\ \gamma_{i,h}(x_i, x_{-i}) = 1 \end{cases} \text{or} \begin{cases} \gamma_{k,h}(x_i, x_{-i}) = 0 \\ \gamma_{k,i}(x_i, x_{-i}) = 1 \\ \gamma_{j,h}(x_i, x_{-i}) = 1 \end{cases} \Rightarrow \gamma_{k,h}(\bar{x}_i, x_{-i}) > \gamma_{k,h}(x_i, x_{-i})$$

while $\gamma_{k,h}(\bar{x}_i, x_{-i}) = \gamma_{k,h}(x_i, x_{-i}) = 0$ or $\gamma_{k,h}(\bar{x}_i, x_{-i}) = \gamma_{k,h}(x_i, x_{-i}) = 1$ otherwise.

Therefore

$$\begin{cases} \mathcal{A}_h(\bar{x}_i, x_{-i}) - \mathcal{A}_h(x_i, x_{-i}) \in \mathbb{R}^n_+, \text{ for all } h \neq i \\ \\ \mathcal{A}_j(\bar{x}_i, x_{-i}) - \mathcal{A}_j(x_i, x_{-i}) \in \mathbb{R}^n_+ \setminus \{0\} \end{cases}$$

and, since $\bar{x}_i \in K_i(x_{-i})$, then x is not a $\mathcal{FP}$ equilibrium and hence the contradiction. So x is two-way connected.

Conversely, assume that $c_{i,j} = c_i$ for all i and for all $j \neq i$ and let x be a two-way connected network without cycles. Then x is Pareto Nash equilibrium in light of the previous proposition. Suppose x is not a $\mathcal{FP}$ equilibrium, this means that there exists a player i and $\hat{x}_i \in K_i(x_{-i})$ such that

$$\begin{cases} \text{(i)}\, \mathcal{A}_h(\hat{x}_i, x_{-i}) - \mathcal{A}_h(x_i, x_{-i}) \in \mathbb{R}^n_+, \text{ for all } h \neq i \\ \text{(ii)}\, \mathcal{A}_j(\hat{x}_i, x_{-i}) - \mathcal{A}_j(x_i, x_{-i}) \in \mathbb{R}^n_+ \setminus \{0\} \quad \text{for some } j \neq i. \end{cases} \tag{7}$$

Since $\hat{x}_i$ does not alter the disutility of players in $I \setminus \{i\}$ with respect to x_i, then condition *ii*) in (7) implies that there exist a player $k \neq j$ such that

$$\begin{cases} \gamma_{k,j}(\hat{x}_i, x_{-i}) = 1 \\ \gamma_{k,j}(x_i, x_{-i}) = 0 \end{cases},$$

but x is two-way connected so $\gamma_{k,j}(x_i, x_{-i}) = 1.$ Hence we get a contradiction and x is a $\mathcal{FP}$ equilibrium.

Remark 5.5. In *ad-networks* the center sponsored star is not an ideal equilibrium since the delation of a link increases the disutility component k_i of the vector payoff of the center i.

Remark 5.6. It is easy to check that the center sponsored star is a strong Pareto Nash equilibrium (Definition 2.3), however many disconnected network structures satisfy this property (see for instance the network in Example 5.1 which is a strong Pareto Nash equilibrium and a disconnected network).

6. Conclusions

In this paper, a new point of view on the problem of social network formation with heterogeneous agents is proposed. Homogeneity, which is usually intended in terms of information or links of equivalent value and identical probabilities of link failure, has been removed in other papers by allowing for different values of information, different costs of connection and different probabilities of link failure. However, in these papers agents are still able to compare a-priori the value of the information coming from different opponents and the probability of link failure is known across links.

Here, it is considered a different kind of heterogeneity in one sided two-way flow networks: agents are not able to compare a-priori the nature and the quality of information coming from the others. For instance, this might be caused by the possibility of link failure with unknown distribution. Therefore there is no a-priori opinion on the relative importance of benefits that each player conjectures to get from connections with the others and, differently from the classical approach, the games of network formation here presented have vector-valued (multicriteria) payoffs so that stable networks are here characterized by the Pareto Nash equilibrium and its refinements.

It turns out that in the so called “*rd-networks*" the results are in line with the previous literature as, for a certain class of parameters, equilibria are characterized by the “two-way connectedness” and “no cycles” properties while an equilibrium refinement (for Pareto Nash), called ideal equilibrium, “plays the role” of the strict Nash equilibrium since it characterizes center sponsored stars.

In the “*ad-networks*”, results are substantially different, since there is no dependence from the parameters of the model and equilibria are characterized only by the “no cycles” property. Moreover, examples show that “stable” networks can be disconnected and that “ideal equilibria” and “strong Nash”-like refinements are ineffective. Hence, in order to characterize the “two-way connectedness” property it is required a generalization to multicriteria games of the “friendliness equilibrium” concept meaning that altruistic motives increase the level of connectedness of the entire network.

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